Lecture 1: Varieties

$k = \bar{k}$ an algebraically closed field.

Def A *variety* over k is a reduced separated finite type scheme over k.

Basic properties

- X scheme is reduced if $\mathcal{O}_X(U)$ is reduced (no nilpotents) for all $U \subset X$ open.

- X scheme is integral if $\mathcal{O}_X(U)$ is an integral domain for all $U \subset X$ open.
  - equivalently, X is reduced and irreducible

Fact integral schemes X have a generic pt η and $\mathcal{O}_{X,\eta}$ is a field
⇝ see exercises

These are affine-local properties

- $f: X \to Y$ - is quasicompact if the preimage of every open in Y is quasicompact
  - locally of finite type if for every open affine $\mathrm{Spec}(B) \subset Y$ and every open affine $\mathrm{Spec}(A) \subset f^{-1}(\mathrm{Spec}(B))$, if the induced morphism
    $B \to A$
    makes A into a finitely generated B-algebra.
  - of finite type if locally of finite type ~~+ qc~~ and quasicompact

(Always) will be working in this category

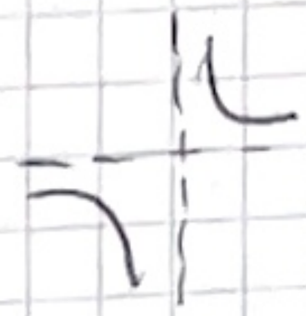
A scheme over k is a scheme ~~$\mathbb{P}$~~ X together with a morphism $X \to \mathrm{Spec}(k)$.
Morphisms of schemes over k are ~~commut~~ morphisms $f: X \to Y$ s.t.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & \mathrm{Spec}(k) \end{tikzcd}$$
commutes

- $\mathrm{Spec}(k)$ is the final object
- Fibre products of schemes over k ~~ex~~ exist.
- Can glue schemes along compatible isomorphisms on overlaps.

→ Dimension and regularity! page 5

Examples

$\mathrm{Spec}\left(k[x,y]/(xy)\right)$  reduced, reducible

$\mathrm{Spec}\left(k[x,y]/(xy-1)\right)$  integral.

$\mathrm{Spec}\left(k[x]/(x^2)\right)$  not reduced.

Separatedness

"AG analogue of Hausdorff"

$f: X \to Y$ is separated if the (relative) diagonal Δ_f is a closed immersion.

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[rr, bend left] \arrow[dr, bend right] & X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

Fact: if X, Y affine, then all morphisms $X \to Y$ are separated.

Cor: Δ_f is always a locally closed immersion

Cor $f: X \to Y$ is separated iff $\Delta_f(X) \subseteq X \times_Y X$ is closed.

Example (line with doubled origins)

$X = U_1 \underset{U_{12}}{\cup} U_2$ $\quad U_i \cong \mathbb{A}^1$ $\quad U_{12} = \mathbb{A}^1 \setminus \{0\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow id$

is not separated over k $\qquad\qquad U_{21} = \mathbb{A}^1 \setminus \{0\}$

Example $\mathbb{P}^1_k = U_0 \cup U_1$ $\quad U_i = \mathbb{A}^1$

$\qquad\qquad\quad \mathbb{A}^1 \setminus \{0\}$

is separated $\qquad U_{01} = \mathbb{A}^1 \setminus \{0\} \longrightarrow U_{10} = \mathbb{A}^1 \setminus \{0\}$

$\qquad\qquad\qquad\qquad\qquad x \longmapsto x^{-1}$

$\mathbb{P}^1$ and line with double origin locally isomorphic $\rightsquigarrow$ separatedness is a global property

Properness

Def • $f: X \to Y$ is universally closed if for all $g: Z \to Y$ the base change $f': X \times_Y Z \to Z$ is closed.

• $f: X \to Y$ is *proper* if it is of finite type, separated, universally closed. AG analogue of cpt

Example • $\mathbb{A}^1_k$ is not proper

• $\mathbb{P}^1_k$ is proper

Fact • Affine schemes$/k$ are proper iff they are zero-dimensional.

• a morphism $f: X \to Y$ is finite iff affine + proper.

Valuative Criteria

Def A valuation on a field K is a map $\nu: K^\times \to (\Gamma, \leq \cdot) = \Gamma$ where Γ is a totally ordered abelian group.

s.th. for all $a, b \in K^\times$ the ~~axiom~~ following axioms hold:

- $\nu(a+b) \geq \min(\nu(a), \nu(b))$
- $\nu(ab) = \nu(a) + \nu(b)$
- $\nu(1) = 0$

• ~~a field with~~ (K, ν) is called a *valued field*

~~and~~

• each valued field (K,ν) has the associated valuation ring

$$R = \{a \in K^\times \mid \nu(a) \geq 0\} \cup \{0\}$$

Example • $\nu: k(t) \to \mathbb{Z}$ w/ valued ring

$f = t^n g \mapsto n$

R DVR $K = \mathrm{Frac}(R)$ $\nu: K \to R$

$f = t^n g \mapsto n$ for t uniform.

• $k((t)) = \left\{ \sum_{n \geq N}^{\infty} a_n t^n \;\middle|\; \begin{matrix} N \in \mathbb{Z} \\ a_n \in k \end{matrix} \right\} \to \mathbb{Z}$

$f = \sum a_n t^n \mapsto \inf\{n \mid a_n \neq 0\}$

Valuative test diagram : VTD

$f: X \to Y$ morphism $\quad (K, \nu, R)$ valued field

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\varphi"] & Y \end{tikzcd}$$

as topological spaces

$\mathrm{Spec}(K) = pt$

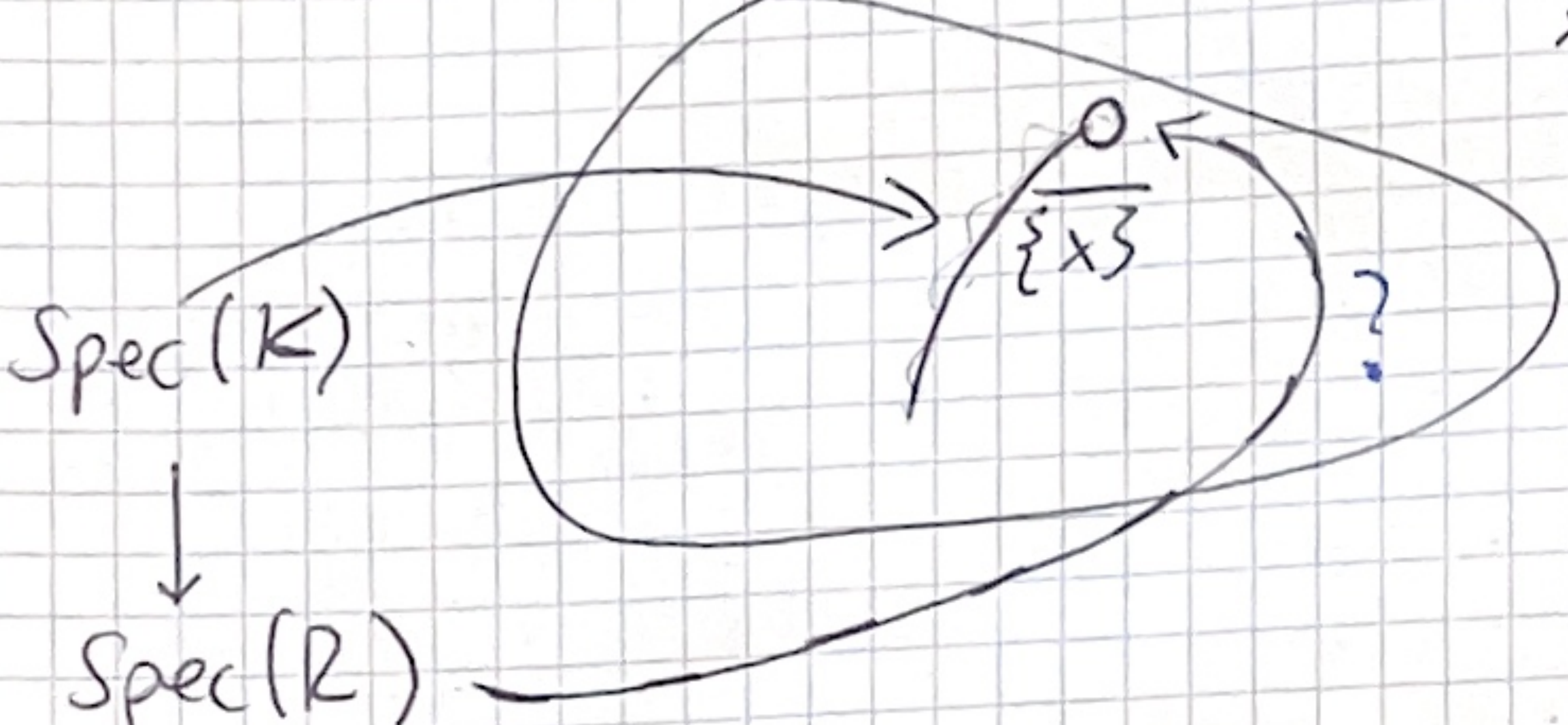
$\mathrm{Spec}(R) \supseteq \{(0), m_R = \{a \in R \mid \nu(a) > 0\}\}$

$=$ if DVR

A ~~an~~ morphism φ is a solution for VTD

Picture for $X \to \mathrm{Spec}(k)$

$x =$ image of $\mathrm{Spec}(K)$



<u>Def</u>. $f: X \to Y$ satisfies <u>the uniqueness part of the valuative criterion</u> if for each VTD there is at most one solution ~~for~~

- $f: X \to Y$ satisfies <u>the existence part of the valuative criterion</u> if for each VTD there exists a solution.

<u>Theorem</u> (Valuative criterion)

$f: X \to Y$ is

(1) <u>separated</u> iff f satisfies the uniqueness part of the VC

(2) <u>universally closed</u> iff f satisfies the existence part of the VC

(3) proper iff f satisfies both parts of the VC

Example $\mathbb{P}^1$ is universally closed:

$$\mathrm{Spec}(K) \xrightarrow{\pi} U_0 \cup U_1$$
$$U_0 = \mathrm{Spec}(k[x]), \quad U_1 = \mathrm{Spec}(k[x^{-1}])$$

~~[illegible]~~ w.l.o.g. $\mathrm{Spec}(K) \to U_0 = \mathrm{Spec}(k[x])$

if $\pi^\flat(x) \in R$ then we have the solution $\mathrm{Spec}(R) \to \mathrm{Spec}(k[x]) \to \mathbb{P}^1$

$$\pi^\flat(x) \longleftarrow\!\shortmid\; x$$

if $\pi^\flat(x) \notin R$, then $\nu^\flat(\pi^\flat(x)) < 0$ (and $\mathrm{Spec}(K) \to U_0 \cap U_1 = \mathrm{Spec}(k[x^{\pm 1}])$)

$\Rightarrow \pi^\flat(x)^{-1} > 0$ and have the solution

$\mathrm{Spec}(R) \to U_1 = \mathrm{Spec}(k[x^{-1}]) \to \mathbb{P}^1$

$$\pi^\flat(x)^{-1} \longleftarrow\!\shortmid\; x^{-1}$$

Dimension

Def X/k irreducible. $\dim(X) := \sup\{\dim(\mathcal{O}_{X,x}) \mid x \in X\}$

$Y \subseteq X$ irreducible closed subset $\mathrm{codim}_Y(X) = \dim(\mathcal{O}_{X,x})$

Recall have bijection $X \xleftrightarrow{1:1} \{Y \subset X \mid Y \text{ irreducible closed}\}$

Regularity / Smoothness

Def X is regular @ $x \in X$ if $\dim(\mathcal{O}_{X,x}) = \dim_{k(x)}(m_x/m_x^2)$

X is regular if it is regular at $x \in X$ for all $x \in X$.

- connected + regular $\Rightarrow$ irreducible
- regular $\Rightarrow$ reduced, normal
- X/k is regular iff it is regular at all k-points.

Theorem (Jacobian criterion)

$U = \mathrm{Spec}(k[x_1, \dots, x_n]/(f_1, \dots, f_m))$

U is regular at a k-point $p \in U$ iff $\mathrm{corank}\left(\left(\frac{df_j}{dx_i}\right)_{i,j}\Big|_p\right) = \dim(U)$

Misc fun:

Prop X connected, reduced, proper $/k = \bar{k}$

$\Rightarrow \Gamma(X, \mathcal{O}_X) = k.$

Pf $\Gamma(X, \mathcal{O}_X) = \mathrm{Hom}_{Sch_k}(X, \mathbb{A}^1_k)$

$\Downarrow$

$\pi : X \to \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$

$\pi(X)^{top}$ connected, $\underset{(\text{in } \mathbb{P}^1!)}{\text{closed}}$ $(\subseteq \mathbb{A}^1) \neq \mathbb{P}^1$

$\Rightarrow \pi(X)^{top} = pt$

$\pi(X)$ reduced $\Rightarrow$ $\pi(X) \cong \underset{\mathrm{Spec}(k)}{pt} \hookrightarrow \mathbb{A}^1$ constant map. $\square$